Moduli Spaces of Riemann Surfaces

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— not quite finished ! —

The Topic.

The topic of this lecture course are the moduli spaces of Riemann surfaces, presented from a topologist's point of view. We will define them and construct models for them, i.e., replace them by a homotopy-equivalent bundle. Our ultimate aim is to study their homology.

Let us denote by $F = F^m_{g,n}$ a connected, compact and oriented surfaces of genus g with n boundary curves and m punctures. We consider the group $\mathrm{Diff}^+ = \mathrm{Diff}^+(F)$ of diffeomorphisms which preserve the orientation, fix the boundary pointwise and possibly permute the punctures. The components of this large group are in almost all cases contractible; thus the interesting information is stored in the group $\Gamma^m_{g,n} = \pi_0(\mathrm{Diff}^+(F^m_{g,n}))$, the so-called *mappping class group.*

Connection with my Seminar on Mapping Class Groups.

We study the mapping class group in my seminar in the summer term 2022. We will learn the basics and many important results, some of geometric nature, some of group-theoretic nature, some of homological nature. See my www-page for the seminar programm. In the lecture course I will sometimes report some facts about mapping class groups without proofs, sometimes I will give proofs.

Moduli Spaces as Objects of algebraic Topology.

Topologists are interested in this group, since $\mathrm{Diff}^+(F^m_{g,n})$ is the structure group for surface bundles. Therefore the classifying spaces $B\mathrm{Diff}^+(F^m_{g,n}) = B\Gamma^m_{g,n}$ is the natural place for characteristic classes of surface bundles. Thus they would like to compute the (co)homolgy of this group, which agrees with the (co)homolgy of the space $B\Gamma)^m_{g,n}$.

The moduli space $\mathfrak{M}^m_{g,n}$ is the space of conformal structures on Riemann surfaces of the give topological type. That for a fixed topological type the complex structure of a surface can vary continuoiusly, is an obvious phenomenon for a complex, differential or algebraic geometer, but a topologist studies usually rigid structures, not structures which do vary continuously. Nevertheless, theses spaces are vary interesting and important, as we see in a moment.

The topology of these spaces is quite subtle and it took a long time until Teichmüller could find a way to define and study it. The Teichmüller theory connects the moduli spaces and the mappping class groups as follows: When $n \geq 1$, then $\mathfrak{M}^m_{g,n}$ has the homotopy type of $B\Gamma^m_{g,n}$ and thus they have the same homology groups. (For $n = 0$, this is at least true rationally.) Clearly, topologists should study them.

The easiest example of a moduli space is the moduli space of all annuli, so $g = 0, n = 2, m = 0$, where the ratio of the outer and inner radius is the only conformal invariant; thus we have a 1-dimensional moduli space $\mathfrak{M}^0_{0,2}$ homoemorphic to the $]1, \infty]$. Another example would be the moduli space of all tori or elliptic curves, so $g = 1, n = 0, m = 0$, given as the quotient of the complex plane by a lattice, where a basis of the lattice, up to scaling, rotation and a Möbius transformation, determines the surface; thus we have a 2-dimensional moduli space $M^0_{1,0}$, homemorphic to a disk (the upper half-plane modulo the action of $\mathrm{SL}_2(\mathbb{R})$.

A Brief History of Moduli Spaces.

It is not easy to say, who first talked about a 'space of conformal structure' of a given surfaces, since already the notion of the underlying homeomorphism or topological type was not yet really developed. But the story begins with the pioneers of complex analysis. Riemann noticed that (for $g \geq 2$) the moduli space $M^0_{g,0}$ has dimension $6g-6$, by simply counting local parameters in certain equations, the solutions of which formed a surface. The topology of these parameter spaces was for a long time very vage, and with the exception of a few examples (like the once mentioned above), many results had only a local nature (like Riemann's formula for the dimension). The global nature of these spaces was largely unknown.

In the meantime, while group theory and topology where developed - and for a major part jointly developed -, the mapping class group $\Gamma^m_{g,n}$ was studied by pioners like Dehn and Nielsen.

It was Teichmüller's great achievement to clarify the topology of the moduli space: he defined the so-called *Teichmüller space* with its metric and defined an action of the mapping class such that the quotient by the action is the moduli space. Most importantly, he showed that the Teichmüller space space is contractible. In all cases where the action is free (e.g., when $n \geq 1$) the moduli space has the homotopy type of $B\Gamma$, classifying space of the mapping class group. This in turn means that (1) the homotopy groups of $\mathfrak{M}^m_{g,n}$ and $\Gamma^m_{g,n}$ agree, in particular, there is only the fundamental group (which is $\Gamma^m_{g,n}$) and all higher homotpy groups vanish; (2) the homology groups of $\mathfrak{M}^m_{g,n}$ agree with the (group) homology of the group $\Gamma^m_{g,n}$; (3) the moduli space is a (non-compact) manifold.

The next decades saw a lot of research on the mapping class group: generators were studied, a presentation was found, allowing the computation the first homology group $H_1(\mathfrak{M}^m_{g,n})$.

Furthermore, the famous homology classes κ_i were defined by Mumford, Morita and Miller; but there was no proof of them being non-triviality. Mumford conjectured, that the stable rational cohomology is a polynomial algebra generated by these classes. Stable(co)homology means here: we consider the limit over the stabilization maps $H_*(\mathfrak{M}^m_{g,n}) \to H_*(\mathfrak{M}^m_{g+1,n})$, defined by attaching a torus with two boundary curves. It is a fundamental result of Harer in 1984, that these homomorphisms are isomorphisms for $* < \frac{2}{3}g$. Furthermore, Harer computed the second homology group $H_2(\mathfrak{M}^m_{g,n})$ and the homological dimension of the moduli spaces.

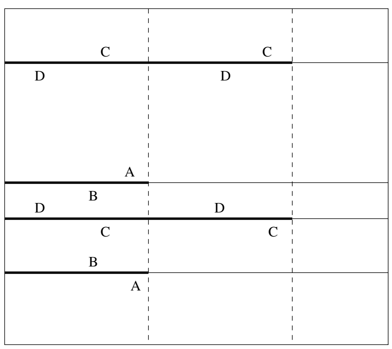
A major breakthrough was the proof of the Mumford conjecture by Madsen & Weiss in 2007. This settles the rational stable homology.

Stable versus Unstable.

As is often the case in topology, the stable situation is easier (or say at least better approachable by general methods) than the unstable situation. A lot of 'noice' just disappears and the picture becomes clearer, when we run to infinity with the genus. We do know very little about the unstable homology (integrally or with other coefficients) of $\mathfrak{M}^m_{g,n}$. But I will give a survey of what is known.

Parallel Slit Domains.

The models for moduli spaces we want to construct are the spaces $\mathfrak{Par}^m_{g,n}$ of parallel slit domains, i.e., configuration spaces of pairs of horizontal slits in one ore several complex planes. This is an old idea, going back to Hilbert in 1909, see [Hi] for the original idea or see [A-B-E] for a quick overview. They share many properties with classical configuration spaces of the plane, but their topology needs quite an amount of combinatorics of symmetric groups. Roughly speaking, the two slits belongig to a pair start with the same real part and run to the left to infinity; this is generic picture. And when two slits (not in a pair) meet, the shorter slit can jump 'over the longer slit to other side of the partner of the longer slit'. Let this suffice as a hint to the sublte topology of these spaces. And instead of too many words here just look at the three figures below to capture the taste of it.



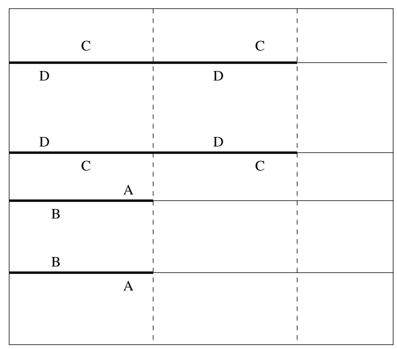


Figure 1: *Two examples of generic parallel slit domains. The letters indicate the gluing of the upper and lower banks of the slit pairs. LEFT EXAMPLE: $g = 1, n = 1, m = 0$ and it is a torus with one boundary curve. The Figures 2 und 3 below show the making of this surface in two steps. RIGHT EXAMPLE: $g = 0, n = 1, m = 2$ and it is a twice punctured disc.*

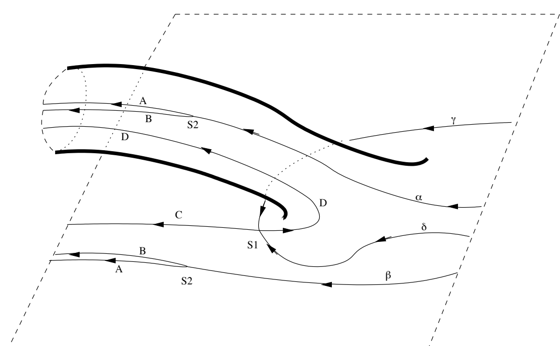


Figure 2: *The surface for Figure 1 (left) is half-way finished.*

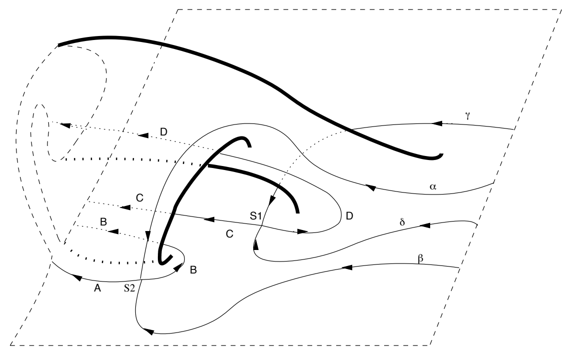


Figure 3: *This is the finished surface.*

Homology Operations.

Special attention will be paid to homology operations on the entire family of moduli spaces. Take two Riemann surfaces both with say one boundary curve. Then we can attach a pair-of-pants with one leg to each of the boundaries and keep the waist as the boundary of the new amalgamated surface. One imagines the pair-of-pants best as the unit disk $\mathbb{D}$ in the plane with two disjoint disks in its interior removed; the outer boundayry is the waist, two other boundaries are the legs. Attach the two surfaces putting them above the plane in 3-space, just having their boundaries touch the two interior boundaries inside the disk. If we fix the pair-of-pants, the construction gives us a product

$$\mu\colon \mathfrak{M}^{m_1}_{g_1,1} \times \mathfrak{M}^{m_2}_{g_2,1} \longrightarrow \mathfrak{M}^{m_1+m_2}_{g_1+g_2,1} \quad . \tag{0.1}$$

And thus we get for the homology groups a product

$$H_i(\mathfrak{M}^{m_1}_{g_1,1}) \otimes H_j(\mathfrak{M}^{m_2}_{g_2,1}) \longrightarrow H_{i+j}(\mathfrak{M}^{m_1+m_2}_{g_1+g_2,1}) \quad , \tag{0.2}$$

called *Pontrjagin product.*

But what, if we do not fix the pair-of-pants, but use the 'space of all pair-of-pants', namely the space of any two interior disjoint disks removed ? Clearly, this space is equivalent to the (unordered) configuration space $C_2(\mathbb{D})$ of two 'distinct, but indistinguishable' points (as a physicist would say). It is equivalent to a circle: just let the two smaller disks rotate around each other. Explicitly, we see an operation of $C_2(\mathbb{D})$ on a product of two moduli spaces

$$M\colon C_2(\mathbb{D}) \times \mathfrak{M}^{m_1}_{g_1,1} \times \mathfrak{M}^{m_2}_{g_2,1} \longrightarrow \mathfrak{M}^{m_1+m_2}_{g_1+g_2,1} \quad . \tag{0.3}$$

Each point $C_2(\mathbb{D})$ gives us a multiplication as above, so we have not one, but a space full of μ's. True, any two points induce the same Pontrjagin product (since the space is connected), but using the rotation parameter 'to integrate over' we can define a 'higher product'

$$[\quad,\quad]\colon H_i(\mathfrak{M}^{m_1}_{g_1,1}) \otimes H_j(\mathfrak{M}^{m_2}_{g_2,1}) \longrightarrow H_{i+j+1}(\mathfrak{M}^{m_1+m_2}_{g_1+g_2,1}), \tag{0.4}$$

raising the sum of degrees by one; it is called the *Browder bracket.* Actually, for any non-trivial homology class in $C_2(\mathbb{D})$ we get a product. And working modulo 2 we can define $\frac{1}{2}[x,x]$, a half-rotation instead of the full rotation $[x,y]$, and called *Dyer-Lashof operation*

$$Q\colon H_i(\mathfrak{M}^m_{g,n};\mathbb{Z}_2) \longrightarrow H_{2i+1}(\mathfrak{M}^{2m}_{2g,2n};\mathbb{Z}_2) \quad . \tag{0.5}$$

One can imagine that we amalgamate not only two, but k surfaces, so the configuration spaces $C_k(\mathbb{D})$ enter the scene and we find a system of operations like (3). One calls the system of all configuration spaces an *operad*, namely the 'little disks operad $\mathcal{C}_2$ in dimension 2', and it operates on the family of moduli spaces. Therefore we need to understand the homolgy groups of configuration spaces.
Note that in the special case $g = 0$ the moduli space of an m-fold punctured disk $\mathfrak{M}^m_{0,1}$ itself is equivalent to the configuration space $C_m(\mathbb{D})$; furthermore, this is the classifying space of the m-fold braid group.
There are many spaces on which the little disks operad in dimension d operates, most importantly the d-fold loop spaces $\Omega^d X$ of a space X, and there is a well-developed theory and many homology computations.

The Lecture Course

This lecture course is an experiment insofar the material can not be found in textbooks and is based on many older and younger research articles or my own ideas about theses spaces. However, there is a vast literature on moduli spaces, usually from the point of view of complex geometry, algebraic geometry or differntial geomtry. I will give hints to the literature during the course. So let me be brief here. For the mapping class group I recommend the book [Fa-Ma]. For the moduli spaces there is a vast literature; you can for example consult the seven volumes [HB].
If you want to attent this lecture course, you should be familiar with the basic concepts of Riemann surfaces, not necessarily moduli spaces. We also need good knowledge of advanced concepts of Algebraic Topology. I will do the course in a rather slow pace, at least in the beginning.

References

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